Parameter identification in ARMA-processes in the presence of regular but incomplete sampling

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PARAMETER IDENTIFICATION IN ARMA PROCESSES IN THE PRESENCE OF REGULAR BUT INCOMPLETE SAMPLING

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Abstract. We discuss the parameter identification of multivariate AR (1) models and of univariate ARMA (2, 1) and AR (2) models if the variables in the model are observed every mth period where m is some integer greater than unity. The results indicate that the models will often not be globally identified even if they are locally identified and that the likelihood function can have a large number of local maxima.

Keywords. Identifiability; missing data; ARMA models.

1. INTRODUCTION

In the analysis of time series it is usually assumed that the data consist of observations on the variables in the model for T subsequent time periods that are considered appropriate on *a priori* grounds. This assumption is often not met in applied work because, for example, economic theory suggests a monthly model and only quarterly data are available or one would like to construct a one-period-ahead forecast of some chemical process which is only observed every other period because of cost considerations. We have shown elsewhere (Nijman and Palm, 1987) that if a variable is known to be generated by an autoregressive integrated moving-average (ARIMA) model it is possible to construct minimum mean square error (MSE) one-period-ahead forecasts even if the variable is only observed every mth period, where m is some interger greater than unity. If, for instance, a monthly ARIMA model is assumed, monthly forecasts can be constructed from quarterly data. Moreover we have shown that the forecast error variance of the forecasts based on the regularly sampled data is not necessarily much larger than the error variance associated with forecasts based on complete data. An important question in this respect is whether the one-period (e.g. monthly) ARIMA model can be identified from regularly sampled (e.g. quarterly) data, i.e. whether the parameters of the one-period model can be uniquely determined from the sample information. Parameter identification is also required if, for instance, an ARIMA model is used to construct approximations for the missing observations as conditional expectations given the available sample information, as suggested by Harvey and Pierse (1984) among others. If the model is identified, efficient parameter estimates can be obtained, for example, using the fact that the observed data series is

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generated by a (possibly constrained) autoregressive moving-average (ARMA) process as will be shown below. The identification and estimation problem might then be viewed as a two-stage problem, of determining the coefficients of the model for the observations and then trying to solve for the underlying parameters. This approach will be adopted in the remainder of this paper.

Some results for the identification of ARIMA models if the variables are only observed every mth period were obtained by Telser (1967) and by Palm and Nijman (1984). The latter authors concentrated on conditions for local identification and showed that $q \leq p + d$ is a necessary condition for identification of a regularly sampled ARIMA (p, d, q) process observed every mth period. Additional *a priori* knowledge will often be required for global identification even if the model is locally identified. Robinson (1980), among others, considers the identification of a univariate continuous-parameter stationary process with rational spectral density which is sampled at times which themselves form a stationary point process.

In this paper we consider the case where the process has a discrete time parameter and is observed every mth period. The global identification conditions are discussed in detail for a multivariate AR(1) model in Section 2 and for univariate ARMA(2,1) and AR(2) models in Section 3.

We conclude that many observationally equivalent locally identified models can easily arise when the observations are incomplete and that, even if the model is globally identified (i.e. if the likelihood function has an overall maximum), the likelihood function will often contain several local maxima.

2. THE MULTIVARIATE AR(1) MODEL

In this section we consider the global identification of a first-order vector autogressive process for which all the variables in the model are observed every mth period only. The global identification criteria for the scalar AR(1) model have already been discussed by Palm and Nijman (1984). The multivariate AR(1) model is as follows:

$$y_t = \Pi y_{t-1} + \varepsilon_t \tag{1}$$

where y_t is a $K \times 1$ vector of variables observed for $t \in T_m = \{m, 2m, \ldots, T\}$, where without loss of generality T is assumed to be a multiple of m. We assume that the eigenvalues of Π lie inside the unit circle and that the ε_t are independent and normally distributed vectors with mean zero and covariance matrix Σ. Of course, the assumption of Gaussianity restricts the possibilities of achieving identification because higher-order moments could contain identifying information if Gaussianity does not hold (e.g. Kapteyn and Wansbeek, 1983). Because the pseudo-maximum likelihood estimators which impose Gaussianity are consistent in these models without Gaussian assumptions, however, the results which we obtain can be used to assess the consistency of these estimators even if Gaussianity does not hold.

If y_t is observed every mth period only, the data-generating process (DGP) is given by

$$y_t = \Pi^m y_{t-m} + u_t \tag{2}$$

where u_t is independent normal and has zero mean and covariance matrix Ω defined by $\Omega = \Sigma_{i=0}^{m-1} \Pi^i \Sigma \Pi^{i\prime}$, where $\Pi^0 = I$. The question is whether Π and Σ can be obtained from the parameters of the DGP (Π^m, Ω). Once Π is known, Σ can be determined from Ω. Therefore we concentrate on the identification of Π.

If the eigenvalues of Π are all distinct we can write (e.g. Rao, 1968) $\Pi = P \Delta P^{-1}$, where Δ is the diagonal matrix with its ith diagonal element equal to the ith eigenvalue of Π and P is the corresponding matrix of eigenvectors. From the above we have that $\Pi^m = P \Delta^m P^{-1}$, so that (if the mth powers of all eigenvalues are distinct) the eigenvalues of Π and the mth powers of their eigenvalues are identified. Compare this with the corresponding result for differential equation models given by Phillips (1973).

If all eigenvalues are real and m is odd, the eigenvalues and Π are identified. If all eigenvalues are real and m is even, 2^K possibilities remain where, as before, K is the dimension of the model. However, the critique by Hansen and Sargent (1983) of the results obtained by Phillips (1973) for the model in continuous time applies here as well: only those matrices Π^* that satisfy $\Pi^{*m} = \Pi^m$ cannot be distinguished from Π for which the corresponding solution Σ^* of $\Omega = \Sigma_{i=0}^{m-1} \Pi^{*i} \Sigma^* \Pi^{*i\prime}$ is positive definite. If m is even, this condition is satisfied by (Π, Σ) and $(-\Pi, \Sigma)$ at least so that the model is not globally identified, as in the univariate case.

If some eigenvalues of Π are complex, the number of solutions can be even larger than 2^K, as $\lambda^{*m} = \lambda^m$ has m solutions in the set of complex numbers. If we ignore the restriction implied by the positive definiteness of Ω and note that complex eigenvalues have to be in conjugate pairs, the number of solutions is at most $m^{K/2}$ if K is even and $2m^{(K-1)/2}$ if K is odd. There is no guarantee that the argument of Hansen and Sargent (1983) will reduce this often considerable number of observationally equivalent models as Σ^* will be close to Ω and therefore positive definite if the eigenvalues of Π are sufficiently small in absolute values. These results could again be compared with the results for the differential equations equivalent to (2) where the number of observationally equivalent models can be infinite even if all eigenvalues are distinct (see Hansen and Sargent, 1983).

In order to illustrate the above results, consider a bivariate AR(1) model and define the matrices

$$P_1 = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.6 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0.7 & -0.3 \\ 0.3 & 0.6 \end{bmatrix} \quad P_3 = \begin{bmatrix} 0.41 & 0.64 \\ 0.64 & 0.20 \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad S_2 = \begin{bmatrix} 1 & -3 \\ -3 & 10 \end{bmatrix}.$$

TABLE I

THE NUMBER OF OBSERVATIONALLY EQUIVALENT MODELS IN THE BIVARIATE AR (1) MODEL (1) FOR VARIOUS CHOICES OF Π, Σ AND m

	$\Pi = P_1$			$\Pi = P_2$			
	$m = 2$	$m = 3$	$m = 4$	$m = 2$	$m = 3$	$m = 4$	$m = 12$
$\Sigma = S_1$	4	1	4	2	3	4	12
$\Sigma = S_2$	2	1	2	2	1	2	4

In Table I we present the number of observationally equivalent models for several values of the number of periods between two observations and for the case where Π and Σ coincide with P_1 or P_2 and S_1 or S_2 respectively. Note that P_1 has real eigenvalues while those of P_2 are complex and that Σ is varied to illustrate the impact of the condition that Σ^* should be positive definite.

It is evident from Table 1 that the number of observationally equivalent models can be large. Moreover, non-trivial *a priori* information will often be required to eliminate all but one of the observationally equivalent models. If $\Pi = P_1$, $\Sigma = S_1$ and $m = 2$ or $m = 4$, the four observationally equivalent models are (P_1, S_1), $(-P_1, S_1)$, (P_3, S_1) and $(-P_3, S_1)$ as can be easily checked. The choice between the first and third of these models requires much more prior information than just the sign of a single coefficient as is the case in the scalar AR(1) model considered by Palm and Nijman (1984). Note that only $(-P_1, S_2)$ is observationally equivalent to (P_1, S_2) if $m = 2$ or $m = 4$, because the models with $\Pi = P_3$ are excluded by the requirement that Σ^* should be positive definite.

Until now we have assumed that the eigenvalues of Π^m are all distinct. In this case the parameters of Π and Σ are locally identifiable although the number of equivalent solutions can be large. If not all eigenvalues are distinct, local identifiability is no longer guaranteed. If, for example, Π is a 2×2 diagonal matrix where the (1,1) and (2,2) elements are ρ and $-\rho$ respectively, we can readily check that, for

$$\Pi^* = \rho \begin{bmatrix} \sqrt{1-bc} & b \\ c & -\sqrt{1-bc} \end{bmatrix},$$

$\Pi^{*2} = \Pi^2$ for all real numbers b, c such that $bc \leq 1$. Therefore there are points in the parameter space arbitrarily close to the true parameter value that are observationally equivalent to it if $K = 2$ and y_t is observed every second period only, and so the model is not locally identified. The discussion of the general case where the eigenvalues are not necessarily distinct is omitted as it is hampered (here and in the continuous-time equivalent) by the fact that the decomposition $\Pi = P\Delta P^{-1}$ no longer applies.

In applied work we have to choose one of the equivalent solutions (Π^*, Σ^*). As illustrated in Table I, this choice requires far more *a priori* information than in the univariate AR(1) case where knowledge of the sign of the autoregressive parameter is sufficient.

3. UNIVARIATE ARMA(2,1) AND AR(2) MODELS

In the previous section we have shown that the number of models which are observationally equivalent to a multivariate AR(1) model can be large if all variables in the model are observed every mth period only. We could consider a large number of generalizations of this, for example, to cases where subsets of the variables are always observed or to higher-order AR or ARMA models. In this section we consider the extension to AR(2) and ARMA(2,1) models only, restricting ourselves moreover to the univariate case. As these models can of course be written in the vector AR(1) representation discussed in the previous section, we might expect the results obtained there to have direct implications for the models to be considered in this section. Unfortunately, however, in the vector AR representation of the higher-order univariate models the assumption made in the previous section that all the variables in the state vector are observed every mth period no longer holds.

Consider the univariate ARMA(2,1) model

$$\rho(L)y_t = \phi(L)\varepsilon_t \tag{3}$$

where $\rho(L) = 1 - \rho_1 L - \rho_2 L^2$, $\phi(L) = 1 + \phi L$ and $\varepsilon_t \sim \text{IN}(0, \sigma_\varepsilon^2)$ and assume that y_t is observed for $t \in T_m$ only. The roots of $1 - \rho_1 z^{-1} - \rho_2 z^{-2} = 0$ are denoted by α_1 and α_2 respectively and it is assumed that $|\alpha_i| < 1$ and $\alpha_i \neq -\phi$ $(i = 1, 2)$.

Multiplication of

$$(1 - \alpha_1 L)(1 - \alpha_2 L)y_t = (1 + \phi L)\varepsilon_t \tag{4}$$

by $s_m(\alpha_1 L)s_m(\alpha_2 L)$, where $s_m(\alpha L) = 1 + \alpha L + \alpha^2 L^2 + \ldots + \alpha^{m-1} L^{m-1}$ yields

$$(1 - \alpha_1^m L^m)(1 - \alpha_2^m L^m)y_t = s_m(\alpha_1 L)s_m(\alpha_2 L)(1 + \phi L)\varepsilon_t. \tag{5}$$

If we assume for simplicity that (5) contains no common factors, the observations are generated by the ARMA(2,1) model

$$(1 - \psi_1 L^m)(1 - \psi_2 L^m)y_t = (1 - \omega L^m)\nu_t \tag{6}$$

where ν_t is a white noise for $t \in T_m$. Its variance σ_ν^2 and the moving-average parameter ω can be obtained by solving the moment equations subject to the condition that $|\omega| < 1$. As an illustration, we first discuss the case when $m = 2$. Then

$$\psi_i = \alpha_i^2 \tag{7a}$$

$$\omega\sigma_v^2 = \{\alpha_1\alpha_2 + \phi\alpha_1 + \phi\alpha_2 + \phi\alpha_1\alpha_2(\alpha_1 + \alpha_2 + \phi)\}\sigma_\varepsilon^2 \tag{7b}$$

and the first-order autocorrelation

$$\frac{\omega}{1+\omega^2} = \{\alpha_1\alpha_2 + \phi\alpha_1 + \phi\alpha_2 + \phi\alpha_1\alpha_2(\alpha_1 + \alpha_2 + \phi)\} \times \{1 + (\alpha_1 + \alpha_2 + \phi)^2 + (\alpha_1\alpha_2 + \phi\alpha_1 + \phi\alpha_2)^2 + \phi^2\alpha_1^2\alpha_2^2\}^{-1}. \tag{7c}$$

The identification conditions of an ARMA(2,1) model observed every second period can easily be checked from (7). The special case with $\alpha_2 = 0$ (i.e. the ARMA(1,1) model) has already been considered by Palm and Nijman (1984). For the ARMA(2,1) model (3), if the standard conditions for identification of $(\psi_1, \psi_2, \omega, \sigma_v^2)$ in (6) are satisfied, $|\alpha_1|$ and $|\alpha_2|$ can be identified from ψ_1 and ψ_2 which yields four admissible solutions for α_1 and α_2. For every admissible solution for (α_1, α_2) the corresponding value for ϕ will have to satisfy (7c). If $\phi = \bar{\phi}$ satisfies this moment equation, so will $\phi = \bar{\phi}^{-1}$ as can easily be verified. Expression (7c) yields a quadratic equation with a unique value for ϕ within the unit circle for every choice of (α_1, α_2). Subsequently a unique value for σ_ε^2 can be obtained from (7b). Four ARMA(2,1) models, corresponding to the different solutions to $\alpha_i^2 = \psi_i$ $(i = 1, 2)$, are observationally equivalent but they are locally identified. If cancelling of factors in (6) is not excluded, the model may not even be locally identified, although the necessary condition for identification given by Palm and Nijman (1984), which requires that the number of moving-average parameters is not larger than the number of autoregressive parameters, is clearly satisfied.

If $\rho_1 = 0$ in (3), which implies that $\alpha_1 = -\alpha_2$, the observations y_t, $t \in T_2$, are generated by $(1 - \rho_2 L^2)y_t = v_t$ with $\sigma_v^2 = (1 + \phi^2)\sigma_\varepsilon^2$. Obviously, the parameters ϕ and σ_ε^2 cannot be identified from σ_v^2.

Returning now to the general case where m is not necessarily equal to 2, we see that (6) implies that if m is odd and if α_1 and α_2 are real and unequal, the parameters α_1 and α_2 can be obtained from ψ_1 and σ_2. The only admissible solution of $\lambda_1^m = \psi_1$ and $\lambda_2^m = \psi_2$ is $(\lambda_1, \lambda_2) = (\alpha_1, \alpha_2)$. The parameters ϕ and σ_ε^2 can then be obtained from ω and ρ_v^2. The model is therefore globally identified in this case. If, in contrast, m is even and α_1 and α_2 are real with $|\alpha_1| \neq |\alpha_2|$, four solution pairs (λ_1, λ_2) are in agreement with (ψ_1, ψ_2). We cannot exclude the possibility that all four solutions are compatible with values for ϕ and σ_ε^2 that are also in agreement with ω and σ_v^2, so that four equivalent models exist. If the roots of $\rho(z^{-1}) = 0$ are complex or $\alpha_1^m = \alpha_2^m$, the number of equivalent models may be even larger. In this case all m solutions of $\lambda_1^m = \psi_1$ in the complex plane cannot be rejected as their conjugates are solutions to $\lambda_2^m = \psi_2$. Analogous to the situation discussed before, we cannot in general expect any information on the correct choice of these roots from ω and σ_v^2. The number of observationally equivalent models can therefore be equal to m. The results for the

TABLE II

THE MAXINUM NUMBER OF OBSERVATIONALLY EQUIVALENT MODELS FOR THE ARMA (2,1) MODEL.

Roots of AR polynomial	mth power of α_1 and α_2	m	Maximum number of observationally equivalent models
Real	Unequal	Odd	1
Real	Unequal	Even	4
Complex	Unequal	Odd/even	m
Real/complex	Equal	Odd/even	m

maximum number of observationally equivalent models for the ARMA(2,1) case are summarized in Table II.

Now consider the AR(2) model where it is known that $\phi = 0$ in (4). The observations on y_t, $t \in T_m$, are again generated by (6), and the discussion of possible solutions of $\lambda_1^m = \psi_1$ and $\lambda_2^m = \psi_2$ goes through as before. However, there is a difference in that ω contains information on the choice of the roots. An incorrect choice for these roots can no longer be compensated by an incorrect choice for ϕ. However, the signs of α_1 and α_2 cannot be uniquely determined if m is even, because a simultaneous change in them does not affect ω. Therefore two observationally equivalent models exist for the AR(2) model if m is even and ρ_2 is globally identified. The information on the choice of the roots λ_1 and λ_2 has to come from ω, the value of which is close to zero for all models that were considered. We should therefore not be surprised if local maxima of the likelihood function close to the overall maxima show up for this model in applications where ω has to be estimated from a finite sample. The difference with the ARMA(2,1) model in this respect will be smaller than suggested by large-sample theory.

To illustrate the points made above we present plots of the approximate log likelihood function for an AR(2) model with $(\rho_1, \rho_2) = (1.4, -0.74)$, $T = 50m$, and $m = 2$ and $m = 3$ respectively. More details and other examples are given by Nijman and Palm (1985).

We define $\psi(L^m) = (1 - \psi_1 L^m)(1 - \psi_2 L^m)$, $\omega(L^m) = 1 - \omega L^m$ and $(\psi_1, \psi_2, \omega, \sigma_v^2) = g(\rho_1, \rho_2, \sigma_\varepsilon^2)$. If the sum of squared residuals in the concentrated log likelihood function $L(\hat{\rho}_1, \hat{\rho}_2)$ is replaced by its expectation or its probability limit for given values of $\hat{\rho}_1$ and $\hat{\rho}_2$, $L(\hat{\rho}_1, \hat{\rho}_2)$ is approximated by

$$f(\hat{\rho}_1, \hat{\rho}_2) = \frac{-T[1 + \ln E\{\hat{\psi}(L^m)\psi^{-1}(L^m)\hat{\omega}^{-1}(L^m)\omega(L^m)v_t\}^2]}{2m} \tag{8}$$

where $(\hat{\psi}_1, \hat{\psi}_2, \hat{\omega}, \hat{\sigma}_v^2) = g(\hat{\rho}_1, \hat{\rho}_2, \hat{\sigma}_\varepsilon^2)$. We plot $f(\hat{\rho}_1, \hat{\rho}_2)$ for admissible values $(\hat{\rho}_1, \hat{\rho}_2)$ which are not significantly different from the true parameter values (ρ_1, ρ_2) at which $f(\hat{\rho}_1, \hat{\rho}_2)$ reaches a maximum, and we plot $f(\rho_1, \rho_2) - 3.00$ if $f(\rho_1, \rho_2) - f(\hat{\rho}_1, \hat{\rho}_2) \geq 3.00$ which indicates that the hypothesis $\rho_i = \hat{\rho}_i$ would be rejected at the 5% level. The value $f(\rho_1, \rho_2) - 3.50$ was assigned to parameter points which imply roots of the

autoregressive polynomial inside the unit circle.

For $(\rho_1, \rho_2) = (1.4, -0.74)$ and $m = 2$, the equations $\lambda_1^m = \psi_1$ and $\lambda_2^m = \psi_2$ have two solution pairs (λ_1, λ_2), $(0.7 + 0.5i, 0.7 - 0.5i)$ and $(-0.7 - 0.5i, -0.7 + 0.5i)$, if we neglect solutions where only the roles of λ_1 and λ_2 are interchanged. Both roots imply the same values of ω as

$$\frac{\omega}{1 + \omega^2} = \frac{\lambda_1\lambda_2}{1 + (\lambda_1 + \lambda_2)^2 + \lambda_1^2\lambda_2^2}.$$

Discrimination between the models with $(\rho_1, \rho_2) = (1.4, -0.74)$ and $(\rho_1, \rho_2) = (-1.4, -0.74)$ is therefore not possible as is evident from Figure 1. If $m = 3$, there are three solutions for (λ_1, λ_2), $(0.7 + 0.5i, 0.7 - 0.5i)$, $(-0.78 + 0.36i, -0.78 - 0.36i)$ and $(0.08 - 0.85i, 0.08 + 0.85i)$. The value of ω can now be derived from

$$\frac{\omega}{1 + \omega^2} = \frac{\lambda_1^2\lambda_2 + \lambda_2^2\lambda_1 + \lambda_1^2\lambda_2^2}{1 + (\lambda_1 + \lambda_2)^2 + (\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)^2 + (\lambda_1^2\lambda_2 + \lambda_1\lambda_2^2)^2 + \lambda_1^4\lambda_2^4}$$

so that $\omega = -0.35$, $\omega = 0.27$ and $\omega = -0.11$ respectively. In large samples the different values of ω can be used to select the correct model, which is globally identified. In small samples the information content of ω will be small and local maxima show up for the three solutions for (λ_1, λ_2) in Figure 2. The maxima in that figure are in good agreement with the values of (ρ_1, ρ_2) implied by the (λ_1, λ_2) solutions which are $(1.4, -0.35)$, $(-1.56, -0.35)$ and $(0.16, -0.35)$ respectively.

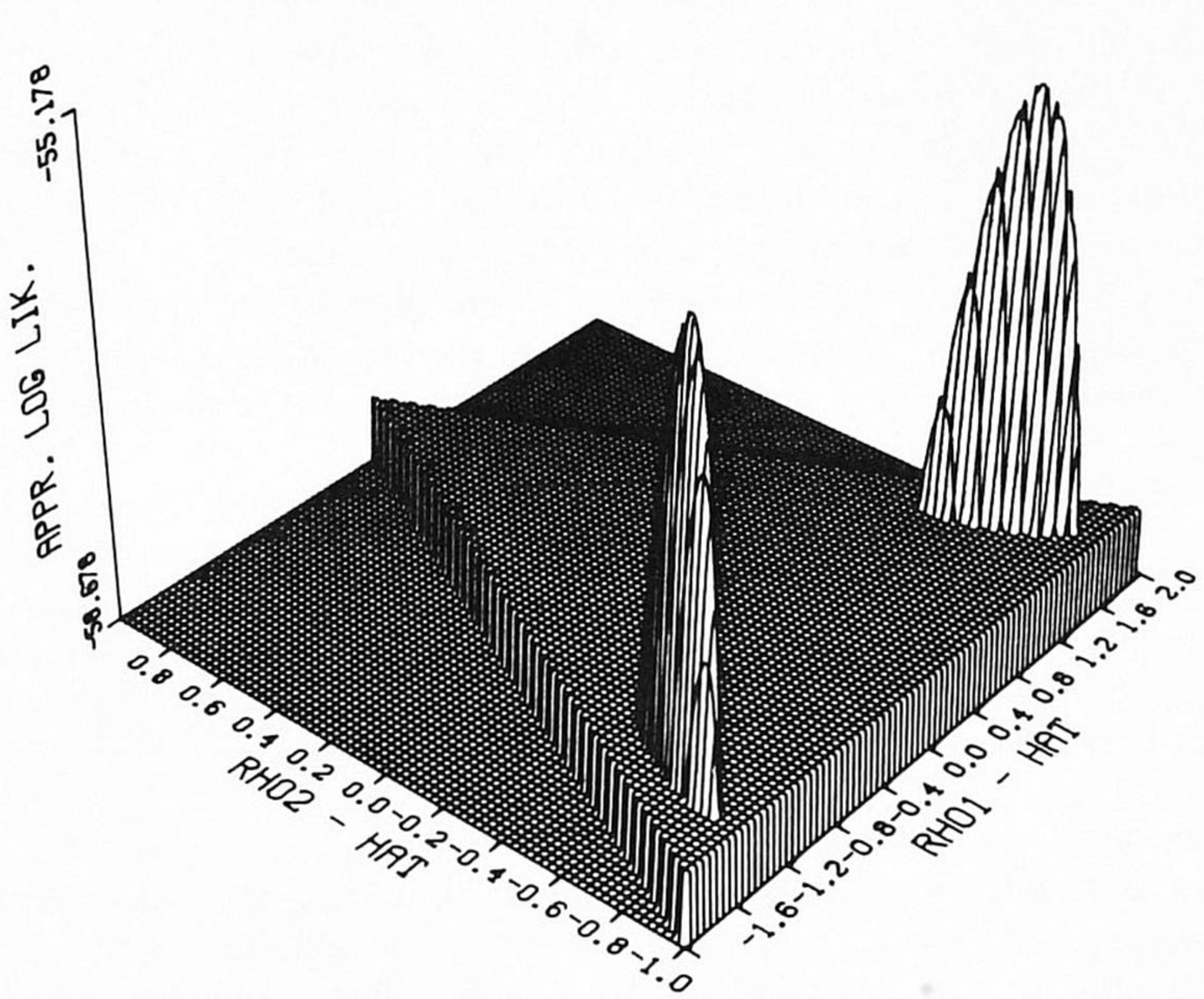

FIGURE 1. Approximate log likelihood function for an AR (2) model with $\rho_1 = 1.4$, $\rho_2 = -0.74$; $m = 2$ and $T = 100$.

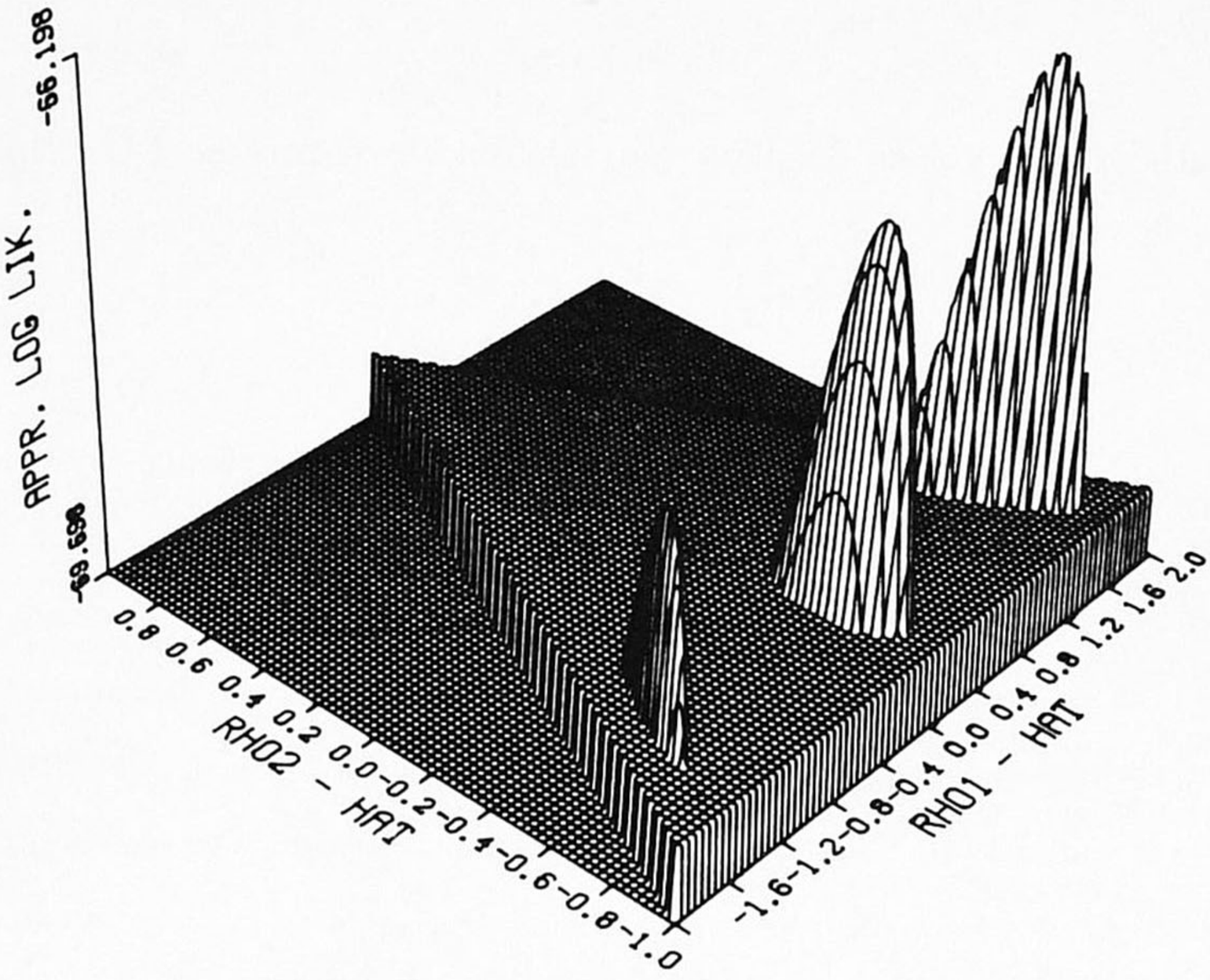

FIGURE 2. As for Figure 1 but with $m = 3$ and $T = 150$.

Extensions of the above discussion to higher-order ARMA models or observations on flow variables, for example, are straightforward. The number of observationally equivalent models can become large, even if the model is locally identified, especially if the autoregressive polynomial contains complex roots.

4. CONCLUSIONS

In this paper we analysed the identification of multivariate AR(1) models and univariate ARMA(2,1) and AR(2) models if the variables in the model are observed every mth period, where m is some integer greater than unity. For a K-variate AR (1) process the number of observationally equivalent locally identified models can be as large as $m^{K/2}$. For the univariate ARMA(2,1) model m observationally equivalent locally identified models can easily arise, while for the univariate AR(2) process the number of observationally equivalent models will usually be small but the likelihood function will often contain several local maxima as illustrated in Section 3. For empirical work our results imply that often additional *a priori* information will be required to identify the parameters and that it is very important to check whether a global maximum has been reached when iterative optimization routines are used to maximize the log likelihood function. However, in a finite sample it cannot be guaranteed that the selection of a solution which may be a long way from the true parameter value will be avoided.

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